Lecture 21

11.1 - Sequences

Def: A sequence is a list of numbers written in a particular order:

$$\{a_1, a_2, a_3, \cdots\}$$

A sequence can be a finite or infinite list. The n^{th} term of the sequence is the n^{th} number in the list.

Ex: In the sequence $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\}$, what is the 2^{nd} term? the 6^{th} term? the n^{th} term?

Sol: $a_2 = \frac{1}{2}$, $a_6 = \frac{1}{6}$, $a_n = \frac{1}{n}$

There are several ways to represent a sequence. Common ones are: (for this example)

$$\left\{-1, \frac{1}{4}, \frac{-1}{9}, \frac{1}{16}, \cdots\right\}, \quad a_n = \frac{(-1)^n}{n^2}, \quad \left\{\frac{(-1)^n}{n^2}\right\}, \quad \left\{\frac{(-1)^n}{n^2}\right\}_{n=1}^{\infty}$$

(Note: There may not always be a formula for the n^{th} term!)

Ex: Find a formula for the n^{th} term of the sequence: $\left\{1, -\frac{1}{4}, \frac{1}{16}, -\frac{1}{64}, \frac{1}{256}, \cdots\right\}$

Sol: The sequence switches $+ - + - \cdots$ so we need a $(-1)^{n-1}$ (or we can use $(-1)^{n+1}$). The denominators are $1, 4, 16, 64, 256, \cdots$, or, $4^0, 4^1, 4^2, 4^3, 4^4, \cdots$, thus the n^{th} term is

$$\boxed{a_n = \frac{(-1)^{n-1}}{4^n}}$$

Graphing a Sequence

To graph a sequence $\{a_1, a_2, a_3, \cdots\}$, we think of it as a function from positive integers to real numbers, and plot the points $(1, a_1), (2, a_2), (3, a_3), \cdots, (n, a_n), \cdots$

What we get is a plot of dots, one over each positive integer.

See mathematica code for examples.

Ex 1: $a_n = \frac{n+1}{n}$ Ex 2: $a_n = \frac{\cos(\frac{n}{4})}{n}$ Ex 3: $a_n = n \sin \frac{n}{4}$

Limit of a Sequence

Notice in examples 1 & 2 that the points start clustering towards a specific value, or getting nearer to a horizontal line/asymptote. This is because these sequences are <u>converging</u> to a value L and this line is $y = L$. Example 3 is an example of a <u>divergent</u> sequence.

<u>Def</u>: A sequence $\{a_n\}$ has the <u>limit</u> L and we write

$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \longrightarrow L \text{ as } n \to \infty$$

if we can make the terms of a_n as close to L as we like by taking n sufficiently large. More precisely, if $\forall \varepsilon > 0$ $\exists$ integer $N > 0$ such that for $n > N$, $|a_n - L| < \varepsilon$. A sequence is <u>convergent</u> if it has a limit, and <u>divergent</u> otherwise.

Theorem

If there is a function $f(x)$ such that $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ for integers n, then

$$\boxed{\lim_{n\to\infty} a_n = L}$$

Ex: Show that the limit of $a_n = \frac{n+1}{n}$ is indeed 1.

Sol: Here $f(n) = \frac{n+1}{n}$, and so

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty}\left(1 + \overset{\to 0}{\frac{1}{n}}\right) = \boxed{1}$$

By this theorem, we can use all of our usual techniques for limits to find limits of sequences, such as L'Hôpital's rule:

Ex: Is the sequence $\left\{\frac{n^2}{4^n}\right\}_{n=1}^{\infty}$ convergent?

Sol: $$\lim_{n\to\infty} \frac{\overset{\to\infty}{n^2}}{\underset{\to\infty}{4^n}} \overset{H}{=} \lim_{n\to\infty} \frac{\overset{\to\infty}{2n}}{\underset{\to\infty}{4^n \ln 4}} \overset{H}{=} \lim_{n\to\infty} \frac{2}{4^n(\ln 4)^2} = 0$$

$\boxed{\text{Yes, it converges to 0.}}$

There are two ways a sequence can diverge: it can diverge by not approaching any value (as in Example 3), or it can diverge to infinity, e.g., $a_n = 2^n$.

All the usual rules for limits apply to sequences too:
If $\{a_n\}$ and $\{b_n\}$ are convergent sequences, and c is any constant:

ⓐ $\lim\limits_{n\to\infty} (a_n \pm b_n) = \lim\limits_{n\to\infty} a_n \pm \lim\limits_{n\to\infty} b_n$

ⓑ $\lim\limits_{n\to\infty} c a_n = c \lim\limits_{n\to\infty} a_n$

ⓒ $\lim\limits_{n\to\infty} (a_n b_n) = \left(\lim\limits_{n\to\infty} a_n\right)\left(\lim\limits_{n\to\infty} b_n\right)$

ⓓ $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}$ (as long as $\lim\limits_{n\to\infty} b_n \neq 0$)

ⓔ $\lim\limits_{n\to\infty} a_n{}^p = \left(\lim\limits_{n\to\infty} a_n\right)^p$ if $p>0$ and $a_n>0$.

ⓕ If $\lim\limits_{n\to\infty} a_n = L$ & $f(x)$ is continuous

$$\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right) = f(L)$$

Ex: Is the sequence convergent or divergent?

$$\left\{\tan\left(\frac{2n\pi}{1+8n}\right)-\frac{n}{1+n^2}\right\}_{n=1}^{\infty}$$

If it converges, what is the limit?

$$\lim_{n\to\infty}\left[\tan\left(\frac{2n\pi}{1+8n}\right)-\frac{n}{1+n^2}\right]=\lim_{n\to\infty}\tan\left(\frac{2n\pi}{1+8n}\right)-\lim_{n\to\infty}\frac{n}{1+n^2}$$

$$=\tan\left(\lim_{n\to\infty}\frac{2n\pi}{1+8n}\right)-0=\tan\left(\frac{2\pi}{8}\right)=\tan\left(\frac{\pi}{4}\right)=\boxed{1}$$

The Squeeze Theorem

If $a_n \le b_n \le c_n$ for $n \geq n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then

Ex: Find the limit of $a_n = \dfrac{\cos\left(\frac{n}{4}\right)}{n}$.

Sol: Since $-1 \le \cos\left(\frac{n}{4}\right) \le 1$ we have: $\dfrac{-1}{n} \le \dfrac{\cos\left(\frac{n}{4}\right)}{n} \le \dfrac{1}{n}$

and since $\lim_{n\to\infty} \pm\frac{1}{n} = 0 \Rightarrow \lim_{n\to\infty} a_n = \boxed{0}$.

Alternating Sequences

For any sequence $\{a_n\}$, we have

$$-|a_n| \le a_n \le |a_n|$$

so, by the squeeze theorem

if $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$

In fact, a sequence with infinitely many positive & negative terms converges <u>if and only if</u> $\lim_{n\to\infty} |a_n| = 0$. In any other case, the limit does not exist.

<u>Ex</u>: Do the sequences

$$\Big\{\underbrace{(-1)^n \frac{2n+1}{n^2}}_{a_n}\Big\}_{n=1}^{\infty} \quad \& \quad \Big\{\underbrace{(-1)^n \frac{2n+1}{n}}_{b_n}\Big\}_{n=1}^{\infty}$$

converge?

<u>Sol</u>:

$|a_n| = \dfrac{2n+1}{n^2}$, $\lim_{n\to\infty} \dfrac{2n+1}{n^2} = 0$, so a_n converges.

$|b_n| = \dfrac{2n+1}{n}$, $\lim_{n\to\infty} \dfrac{2n+1}{n} = 2 \ne 0$, so b_n diverges.

Monotone Sequences

Def: A sequence $\{a_n\}$ is called:

- increasing if $a_n < a_{n+1}$ for all $n \geq 1$
- decreasing if $a_n > a_{n+1}$ for all $n \geq 1$

A monotone sequence is one which is increasing or decreasing.

Def: A sequence $\{a_n\}$ is:

- bounded above if $\exists M$ such that $a_n \leq M$ for $n \geq 1$
- bounded below if $\exists m$ such that $a_n \geq m$ for $n \geq 1$

A sequence is bounded if it is both bounded above and below.

Theorem: Every bounded monotonic sequence is convergent.

Check for Monotonicity: If our sequence can be written $a_n = f(n)$, where f is differentiable, $\{a_n\}$ is

- increasing if $f'(x) > 0$
- decreasing if $f'(x) < 0$

Def: Is the sequence $\{\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \cdots\}$ monotone? bounded? convergent?

Sol: $\sqrt{2} = 2^{1/2}$, $\sqrt{2\sqrt{2}} = 2^{3/4}$, $\sqrt{2\sqrt{2\sqrt{2}}} = 2^{7/8}$, $\cdots$

The powers are $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \cdots$

the n^{th} power is $\frac{2^n-1}{2^n}$, so the sequence's n^{th} term is

$$a_n = 2^{\left(\frac{2^n-1}{2^n}\right)}$$

So, since $\frac{2^n-1}{2^n} < 1$, $a_n \leq 2$ and a_n is always positive, so $a_n \geq 0$, thus a_n is bounded.

Since $\frac{2^{n-1}-1}{2^{n-1}} < \frac{2^n-1}{2^n}$ $\left(\Rightarrow 2(2^{n-1}-1) < 2^n - 1 \Rightarrow 2^n - 2 < 2^n - 1 \checkmark\right)$

we have $2^{\frac{2^{n-1}-1}{2^{n-1}}} = a_{n-1} < a_n = 2^{\frac{2^n-1}{2^n}}$

so a_n is monotone increasing. Since a_n is monotone and bounded, it converges.